RELIABILITY ASSESSMENT FOR HIGHLY RELIABLE SYSTEMS

BY

M. V. JOHNS, JR.

TECHNICAL REPORT NO. 1

NOVEMBER 15, 1975

U.S. ARMY RESEARCH OFFICE

RESEARCH TRIANGLE PARK, NORTH CAROLINA

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Introduction. The purpose of this study is to develop appropriate methods for the assessment of reliability for highly reliable systems, i.e., systems characterized by failure probabilities no larger than 5%. Particular emphasis is given to cases where the probability of system failure during any particular duty cycle is less than 1% and where substantial amounts of subsystem test data are available. Serial systems consisting of several component subsystems are considered and confidence bounds for the mean number of trials between failures based on subsystem failure data are developed. (The mean number of trials between failures (MTBF) is equal to $1/(1-R)$ when R is the system reliability, i.e., the probability of completion of a failure-free duty cycle.) Tables needed for the implementation of the proposed procedure are provided. It is assumed that the failure data available for the component subsystems consists of the numbers of failures observed in a substantial number of independent trials for each subsystem where the numbers of trials may differ for different subsystems. Previously available methods for obtaining confidence bounds for series system reliability have typically required the assumption of exponentially distributed times to failure (see e.g. [3], [10]) or have involved ad hoc approximations (see e.g. [11]). Generally the data are assumed to have been obtained through inverse sampling (Type II censoring), i.e., subsystem testing continues until a predetermined number of failures has been observed—a procedure often difficult to implement in practice (see e.g. [3], [7]). The approach used in the present study is non-Bayesian, thereby avoiding the difficulties associated with subjective

probabilities. The underlying rationale is based on exact methods, although asymptotic approximations are proposed and evaluated.

Consider a system consisting of k subsystems operating independently in a series configuration. Let p_i be the probability that the i^{th} subsystem will fail on any given trial (duty cycle) and suppose that the subsystem is subjected to n_i trials, $i = 1,2,\ldots,k$. We envisage a situation where the p_i's are substantially less than 1% and the n_i's are of the order of several hundred to several thousand. For such cases, if X_i is the number of observed failures for the i^{th} subsystem, then the representation of the distribution of X_i by the Poisson approximation with parameter $\lambda_i = n_i p_i$ is essentially exact. The system reliability is given by

$$R = \prod_{i=1}^{k} (1-p_i) = \prod_{i=1}^{k} \left(1 - \frac{\lambda_i}{n_i}\right).$$

For the values of the p_i's considered here the approximation

$$R \doteq 1 - \sum_{i=1}^{k} \lambda_i/n_i$$

is essentially exact so that to find a lower confidence bound for the mean time between failures = MTBF = $1/(1-R)$, or for the system reliability, R, it suffices to obtain an upper confidence bound for $\sum_{i=1}^{k} \lambda_i/n_i$.

The problem thus reduces to that of finding an upper confidence bound for a linear function of Poisson parameters based on independent Poisson observations $X_1, X_2, \ldots, X_k$. To treat this problem we introduce the parameter

$$\theta = \sum_{i=1}^{k} a_i \lambda_i , \tag{1}$$

where the a_i's are normalized coefficients with $\sum_{i=1}^{k} a_i = 1$ and $a_i > 0$, $i = 1,2,\ldots,k$. The use of a_i's which sum to one facilitates the construction of numerical tables. The connection with the original problem is as follows: For $i = 1,2,\ldots,k$ let

$$a_i = \left[n_i \left(\sum_{i=1}^{k} 1/n_i\right)\right]^{-1} . \tag{2}$$

If θ^* is an upper confidence bound at confidence level $1-\alpha$ for θ given by (1), then the corresponding upper confidence bound for $\sum_{i=1}^{k} \lambda_i/n_i$ is given by $\theta^* \left(\sum_{i=1}^{k} 1/n_i\right)$, and the $1-\alpha$ level confidence statement for the quantity MTBF is

$$\text{MTBF} \geq \left[\theta^* \left(\sum_{i=1}^{k} 1/n_i\right)\right]^{-1} . \tag{3}$$

Because of the additive property of the Poisson distribution, the data for subsystems for which the sample sizes n_i are the same may be combined so that k represents the number of <u>different</u> subsystem sample sizes or the number of <u>distinct</u> a_i's. If all subsystems are subjected to the same number of trials we may take k equal to one and the problem reduces to the familiar one of finding an upper confidence bound for a single Poisson parameter.

The subsystem failure data may be developed through independent testing of the subsystems, or through testing of the complete system with the assignment of failures to the appropriate subsystems. Even in the latter case subsystem sample sizes may differ if subsystems are redesigned during the course of testing so that the trials and failures observed prior to redesign for such subsystems are not relevant to the reliability of the final version of the system.

It is worth noting that the same formal confidence bound problem arises from a somewhat different and more restrictive model involving continuous time sampling and exponentially distributed failure times. Thus, if we assume that the time to failure of the i^{th} subsystem has the exponential probability density function $\mu_i e^{-\mu_i t}$, $\mu_i > 0$, $t > 0$, and if the fixed aggregate testing time for the i^{th} subsystem is given by τ_i and the total number of failures observed is N_i, $i = 1,2,\ldots,k$, then the N_i's which are sufficient statistics have independent Poisson distributions with corresponding parameters $\lambda_i = \mu_i \tau_i$. Such test data would be generated, for example, if a single unit of each subsystem were placed on test and repaired whenever a failure occurred until the total test time for each subsystem reached its corresponding limit, τ_i. The system reliability for a single duty cycle (consisting of operation for a unit interval of time) is given by

$$R = \prod_{i=1}^{k} e^{-\mu_i} = \exp\left(-\sum_{i=1}^{k} \mu_i\right) .$$

If the system is highly reliable the μ_i's will be very small and we have

$$1 - R \doteq \sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} \lambda_i/\tau_i$$

and the confidence bound problem becomes formally the same as the one previously introduced.

In Section 2 the general method for constructing the required confidence bounds is described and optimality properties are discussed. The method is based on the use of an appropriate linear ordering imposed upon the collection of possible sample vectors $\underline{X} = (X_1, X_2, \ldots, X_k)$ and exploits an idea first introduced by Buehler [2]. The use of the tables which have been computed for the cases $k = 1$ and $k = 2$ is illustrated in this section.

In Section 3 approximations are introduced which permit the calculation of the required confidence bounds for data beyond the range of the computed tables. These approximations, which are based on the maximum likelihood ratio (MLR) statistic and the asymptotically equivalent maximum likelihood estimate (MLE) confidence bound, are used to justify a procedure for reducing cases where $k \geq 3$ to the $k = 2$ case. The asymptotic expressions obtained are not, strictly speaking, "large sample" results since it is not the sample sizes (the n_i's) which are assumed to become large but rather the largest of the λ_i's .

The formal proofs of the optimality results of Section 2 and the asymptotic results of Section 3 are deferred to Appendices 1 and 2. Appendix 3 describes the construction of the tables.

2. Confidence Bounds. A frequent practice in dealing with confidence bound problems is to use procedures which are "optimal" in the sense that they correspond to a hypothesis testing procedure which has some desirable property such as being, say, uniformly most powerful unbiased. Such confidence bounds are said to be "uniformly most accurate unbiased" (see [5], pp. 176-180). This approach is relevant only when UMP tests exist and is somewhat questionable in any case, since the criteria of interest in confidence statements such as the expected size of the confidence region have no counterparts in the hypothesis testing context. For the present case, no UMP unbiased or invariant tests of hypotheses concerning linear functions of Poisson parameters exist so that another approach is required. As noted in Section 1, we have elected to exploit an idea introduced by Beuhler [2] which involves the construction of confidence bounds based on suitably generated ordering of the sample data vectors.

Before discussing the specific problem of constructing an upper confidence bound for the parameter θ defined by (1), we consider the following more general situation: Let $P_{\underline{\lambda}}$ represent a family of probability functions indexed by a (vector) parameter $\underline{\lambda}$ lying in a set Λ, and suppose that for each $\underline{\lambda}$, $P_{\underline{\lambda}}$ is defined on the appropriate subsets of a sample space $\mathcal{X}$ whose (possibly vector) generic point is designated by $\underline{x}$ with $\underline{X}$ denoting the corresponding random vector. Let $\theta = \theta(\underline{\lambda})$ represent a real valued function of $\underline{\lambda}$. Suppose further that a linear ordering (denoted by $\succeq$) of the elements of the sample space is specified and that this ordering may be thought of as being generated by a real valued function $\phi(\underline{x})$. I.e., $\underline{x} \succeq \underline{y}$ if and only if $\phi(\underline{x}) \geq \phi(\underline{y})$. We seek an upper $1-\alpha$ level confidence bound $\theta^*(\underline{X})$ for θ which is monotone

in the specified ordering on $\mathcal{X}$. Thus we require

(4) $$P_{\underline{\lambda}}\{\theta \leq \theta^*(\underline{X})\} \geq 1 - \alpha \quad \text{for all} \quad \underline{\lambda} \in \Lambda \text{ , and}$$

(5) $$\underline{x} \succeq \underline{y} \iff \theta^*(\underline{x}) \geq \theta^*(\underline{y}) .$$

To achieve this we proceed as follows: For each real t in the range of $\theta(\underline{\lambda})$ let $S(t) = \{\underline{\lambda}; \theta(\underline{\lambda}) = t\}$.

Assumption A. For each $\underline{x} \in \mathcal{X}$ the function

$$h_{\underline{x}}(t) = \sup_{\underline{\lambda} \varepsilon S(t)} P_{\underline{\lambda}}\{\underline{X} \preceq \underline{x}\}$$

is monotone decreasing in t , and there exists a value $t_\alpha(\underline{x})$ such that

$$h_{\underline{x}}(t_\alpha(\underline{x})) = \alpha ,$$

and the collection of such values contains its greatest lower bound.

Def. For each $\underline{x} \in \mathcal{X}$, $\theta^*(\underline{x})$ = smallest $t_\alpha(\underline{x})$.

The properties of $\theta^*(\underline{x})$ are outlined in the following four propositions, the proofs of which are deferred to Appendix 1:

Proposition 1. Assumption A implies that $\theta^*(\underline{x})$ is monotone in the ordering $(\succeq)$.

Proposition 2. Under Assumption A $\theta^*(\underline{x})$ is a $1-\alpha$ level confidence bound for $\theta(\underline{\lambda})$.

Proposition 3. Under Assumption A, if $\tilde{\theta}(\underline{x})$ is any other $1-\alpha$ level confidence bound for $\theta(\underline{\lambda})$ which is monotone in $(\succeq)$ then $\tilde{\theta}(\underline{x}) \geq \theta^*(\underline{x})$ for all $\underline{x} \in \mathcal{X}$.

Proposition 4. Under Assumption A, $\theta^*(\underline{x})$ is undominated in the sense that if $\tilde{\theta}(\underline{x})$ is any $1-\alpha$ confidence bound, then $\sup_{\underline{y} \leq \underline{x}} \tilde{\theta}(\underline{y}) \geq \theta^*(\underline{x})$ for all $\underline{x} \in \mathcal{X}$.

Proposition 3 guarantees that θ^* is optimal with respect to the given ordering and Proposition 4 insures weak admissibility with respect to any ordering, i.e., there does not exist a $1-\alpha$ level confidence bound $\tilde{\theta}$, say, such that $\tilde{\theta}(\underline{x}) < \theta^*(\underline{x})$ for all $\underline{x} \in \mathcal{X}$. Furthermore, if for every distinct $\underline{x}_1, \underline{x}_2 \in \mathcal{X}$, $\theta^*(\underline{x}_1) \neq \theta^*(\underline{x}_2)$, then for any $1-\alpha$ level confidence bound $\tilde{\theta}$, if $\tilde{\theta}(\underline{x}') < \theta^*(\underline{x}')$ for some $\underline{x}' \in \mathcal{X}$, then $\tilde{\theta}(\underline{x}'') > \theta^*(\underline{x}'')$ for some $\underline{x}'' \in \mathcal{X}$. The choice of the ordering $(\succcurlyeq)$ or, equivalently, the order generating function $\phi(\underline{x})$ remains to be determined. In the present application ϕ will be chosen so as to insure that $\theta^*(\underline{x})$ is reasonable for small true values of $\theta(\underline{\lambda})$ and asymptotically optimal in the usual sense (uniformly most accurate) for large $\theta(\underline{\lambda})$.

Returning to the specific problem under consideration, we recall that $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\theta(\underline{\lambda}) = \sum_{i=1}^{k} a_i \lambda_i$ and $\underline{X} = (X_1, X_2, \ldots, X_k)$ where the X_i's are independent observations and X_i has a Poisson distribution with parameter λ_i, $i = 1,2,\ldots,k$. The verification that Assumption A is satisfied for this case is deferred to Appendix 1.

To find $\theta^*(\underline{x})$ for the case $k=1$ (where all the subsystem sample sizes are equal) we use the natural ordering of $x_1 = x = 0,1,2,\ldots$ and observe that $\theta(\underline{\lambda}) = \lambda_1$ so that $\theta^*(x)$ is simply the value of λ_1 for which $P_{\lambda_1}\{X \leq x\} = \alpha$. This corresponds to the standard UMP one-sided test concerning λ_1, and θ^* coincides with the "classical" $1-\alpha$ level confidence bound for λ_1. Tables for this bound have usually been based on the relationship between the Poisson and the chi-squared distributions. We have used the Poisson distribution directly to compute Tables 1 and 2, which give $\theta^*(x)$ for $\alpha = .10$ and $\alpha = .05$ for values of x between 0 and 149.

For the case $k \geq 2$ (k distinct subsystem sample sizes) the determination of θ^* requires the simultaneous solution of k non-linear equations. To see this we note that for general k

$$(6) \qquad P_{\underline{\lambda}}\{\underline{X} \preceq \underline{x}\} = \sum_{\underline{y} \preceq \underline{x}} P_{\underline{\lambda}}\{\underline{X} = \underline{y}\} \ ,$$

where

$$(7) \qquad P_{\underline{\lambda}}\{\underline{X} = \underline{y}\} = \exp\left\{-\sum_{i=1}^{k} \lambda_i\right\} \prod_{i=1}^{k} (\lambda_i^{y_i}/y_i!) \ .$$

We must maximize (6) under the constraint $\sum_{i=1}^{k} a_i\lambda_i = t$ and obtain $\theta^*(x)$ as the value of t for which the maximum assumes the value α. The constrained maximization may be accomplished by setting

$$\lambda_k = \frac{1}{a_k}\left(t - \sum_{i=1}^{k-1} a_i\lambda_i\right)$$

in (6) and solving simultaneously the $k-1$ equations obtained by setting the partial derivatives of (6) with respect to λ_i , $i = 1,2,\ldots,k-1$, equal to zero, with the k^{th} equation obtained by setting (6) equal to α. This is equivalent to finding the solution $\underline{\lambda}^* = (\lambda_1^*,\lambda_2^*,\ldots,\lambda_k^*)$ with $\lambda_i^* \geq 0$, $i = 1,2,\ldots,k$, of the simultaneous equations

$$\frac{\partial}{\partial\lambda_j} P_{\underline{\lambda}}\{X \preceq \underline{x}\} + \frac{\partial}{\partial\lambda_k} P_{\underline{\lambda}}\{X \preceq \underline{x}\} \frac{\partial}{\partial\lambda_j} \frac{1}{a_k}\left(t - \sum_{i=1}^{k-1} a_i\lambda_i\right) = 0 \ , \qquad j = 1,2,\ldots,k-1$$

or

$$(8) \qquad \frac{\partial}{\partial\lambda_j} P_{\underline{\lambda}}\{X \preceq \underline{x}\} - \frac{a_j}{a_k} \frac{\partial}{\partial\lambda_k} P_{\underline{\lambda}}\{X \preceq \underline{x}\} = 0 \ , \qquad j = 1,2,\ldots,k-1 \ ,$$

and

$$(9) \qquad P_{\underline{\lambda}}\{X \preceq \underline{x}\} = \alpha ,$$

and setting

$$(10) \qquad \theta^*(\underline{x}) = \sum_{i=1}^{k} a_i \lambda_i^* .$$

Observing that

$$(11) \qquad \frac{\partial}{\partial \lambda_j} P_{\underline{\lambda}}\{X = \underline{y}\} = \left(\frac{y_j}{\lambda_j} - 1 \right) P_{\underline{\lambda}}\{\underline{X} = \underline{y}\} ,$$

we may write (8) and (9) more explicitly as

$$(12) \qquad \sum_{\underline{y} \preceq \underline{x}} \left\{ a_k \left(\frac{y_k}{\lambda_k} - 1 \right) - a_j \left(\frac{y_j}{\lambda_j} - 1 \right) \right\} P_{\underline{\lambda}}\{\underline{X} = \underline{y}\} = 0 , \qquad j = 1,2,\ldots,k-1 ,$$

and

$$(13) \qquad \sum_{\underline{y} \preceq \underline{x}} P_{\underline{\lambda}}\{\underline{X} = \underline{y}\} = \alpha$$

where $P_{\underline{\lambda}}\{\underline{X} = \underline{y}\}$ is given by (7).

In **Appendix** 2 it is shown that for large $\gamma = \max(\lambda_1, \lambda_2, \ldots, \lambda_k)$ the asymptotically optimal upper confidence bound for $\theta(\underline{\lambda})$ is given by

$$(14) \qquad \hat{\theta}(\underline{x}) = \sum_{i=1}^{n} a_i X_i + c_\alpha \left(\sum_{i=1}^{n} a_i^2 X_i \right)^{1/2}$$

where c_α is the upper $(100\alpha)^{th}$ percentage point of the standard normal distribution. This bound is based on the asymptotic normal distribution of the maximum likelihood estimate of $\theta(\underline{\lambda})$. We therefore choose to generate the ordering of the sample points $\underline{x}$ by means of the function $\phi(\underline{x}) = \hat{\theta}(\underline{x})$. This guarantees that (i) the bound $\theta^*(\underline{x})$ will be asymptotically optimal and (ii) in the absence of strong prior information concerning relative subsystem reliabilities, $\theta^*(\underline{x})$ will be "reasonable" for small values of γ (and hence for small X_i's) in the sense that the subsystems are treated evenhandedly according to the principle of maximum likelihood. It should be born in mind that only the ordering of the sample points is determined by $\phi(\underline{x})$ and that any monotone function of ϕ would produce the same ordering. Small sample deficiencies in the maximum likelihood bound will not therefore be reflected in the performance of $\theta^*(\underline{x})$.

For the case $k = 2$ (two distinct subsystem sample sizes), Tables 3 and 4 give $\theta^*(\underline{x})$ for $\alpha = .10$ and $\alpha = .05$ for the first 100 ordered values of the sample point $\underline{X} = (X_1, X_2)$ for six representative values of a_1 ($a_1 = .10, .20, .25, .30, 1/3, .40$). Note that a_1 is necessarily less than $1/2$ since $a_1 \leq a_2 = 1 - a_1$ and the case $a_1 = a_2 = 1/2$ reduces to the case $k=1$.

Example 1. Consider a weapons system consisting of five subsystems and suppose that the entire system is subjected to 10,000 trials (duty cycles) with necessary repairs being made after each subsystem failure. Suppose further that subsystem number five is redesigned after 5,000 trials so that only data for the last 5,000 trials of that subsystem are relevant to the reliability of the final system configuration. Suppose that the

observed data are as follows:

Subsystem	Number of Trials	Observed Failures
1	10,000	1
2	10,000	10
3	10,000	5
4	10,000	0
5	5,000	1

Since the first four subsystems were subjected to the same number of trials $(n_1 = 10{,}000)$ we combine their failure data, obtaining $X_1 = 16$. For the remaining subsystem we have $n_2 = 5{,}000$ and $X_2 = 1$. To find a 90% lower confidence bound for MTBF for this system we first note that $a_1 = 1/[10{,}000(1/10{,}000 + 1/5{,}000)] = 1/3$ by (2). From Table 3 we find that the upper bound θ^* for θ corresponding to $a_1 = 1/3$ and $(X_1, X_2) = (16,1)$ is $\theta^* = 8.422$. By (3) we may then assert that at the 90% level of confidence

$$\text{MTBF} \geq 1/[8.422(1/10{,}000 + 1/5{,}000)] = 396 .$$

Example 2. For the system considered in Example 1, suppose that the failure data were the same as those given above except that the redesign of the fifth subsystem was not performed so that its sample size is also 10,000 trials. Since all the sample sizes are equal we are in the case $k=1$ with the total number of failures being $X = 17$. For a 90% confidence bound we note that from Table 1, for $X = 17$ the value of the upper bound θ^* is 23.606. The 90% confidence statement for MTBF then becomes

$$\text{MTBF} \geq 1/[23.606(1/10{,}000)] = 424 .$$

It should be noted that had the number of trials in these examples been as small as 500 to 1000, the corresponding analysis would still be valid since the Poisson approximations for the binomial distributions would still be very close.

3. Approximations. In Appendix 2 the asymptotic theory (for large $\gamma = \max(\lambda_1,\lambda_2,\ldots,\lambda_k)$) of the maximum likelihood ratio procedure for testing hypotheses concerning θ is used to obtain approximate $1-\alpha$ level confidence bounds for $\theta(\underline{\lambda})$. Specifically, if $\hat{\mu}$ is the positive real root less than $1/\max(a_1,a_2,\ldots,a_k)$ of the equation

$$(15) \qquad \chi^2_{1,2\alpha} = 2\sum_{i=1}^{k} X_i \left\{\frac{a_i\hat{\mu}}{1-a_i\hat{\mu}} + \log(1-a_i\hat{\mu})\right\} ,$$

where $\chi^2_{1,2\alpha}$ is the upper $100(2\alpha)^{th}$ percentage point of the chi-squared distribution with one degree of freedom, then the quantity

$$(16) \qquad \tilde{\theta}(\underline{X}) = \sum_{i=1}^{k} a_i X_i/(1-a_i\hat{\mu})$$

is an approximate upper $1-\alpha$ level confidence bound for $\theta(\underline{\lambda})$. This bound is asymptotically optimal and asymptotically equivalent to $\hat{\theta}$ given by (14), but it is more precise than the latter for moderate values of the X_i's. Equation (15) may easily be solved for $\hat{\mu}$ by trial and error approximation using a hand calculator. For the case $k = 2$, $\tilde{\theta}$ is a good approximation to θ^* for values of (X_1,X_2) beyond those tabulated. Comparisons of the M.L.R. approximations with the exact values of θ^* from Tables 3 and 4 for the "largest" pairs (X_1,X_2) (corresponding to $I = 100$) for each case are given in Tables A and B below:

Table A (α=.10, k=2)

	a_1=.10	a_1=.20	a_1=.25	a_1=.3	a_1=1/3	a_1=.4
	X_1=30, X_2=1	X_1=4, X_2=5	X_1=10, X_2=4	X_1=0, X_2=8	X_1=15, X_2=2	X_1=7, X_2=6
$\tilde{\theta}$ (M.L.R.)	5.70	7.59	8.05	8.53	8.65	9.02
θ^* (Table 3)	6.24	7.88	8.36	9.10	8.81	9.18

Table B (α=.05, k=2)

	a_1=.10	a_1=.20	a_1=.25	a_1=.30	a_1=1/3	a_1=.40
	X_1=17, X_2=2	X_1=3, X_2=5	X_1=10, X_2=4	X_1=16, X_2=2	X_1=6, X_2=6	X_1=15, X_2=1
$\tilde{\theta}$ (M.L.R.)	6.53	8.34	8.92	9.23	9.58	9.73
θ^* (Table 4)	7.13	8.70	9.12	9.60	9.77	10.10

The M.L.R. bound always tends to underestimate the true bound, but the approximations are seen to be within 10% of the exact values in all cases and within 5% in most cases. The precision of $\tilde{\theta}$ will, of course, improve further for larger X_i's .

For $k \geq 3$, if some of the X_i's are moderately large, it is reasonable to expect that the M.L.R. bound $\tilde{\theta}$ will give satisfactory results. For $k \geq 3$ and small X_i's a better approximation is required. We observe that the asymptotic bound $\hat{\theta}$ given by (14) depends only on the maximum likelihood estimate of θ , i.e.,

(17) $$\bar{\theta} = \sum_{i=1}^{k} a_i X_i ,$$

and the maximum likelihood estimate of the variance of $\bar{\theta}$, i.e.,

(18) $$\bar{v} = \sum_{i=1}^{k} a_i^2 X_i .$$

This suggests that cases involving $k \geq 3$ may be reduced approximately to corresponding cases with $k = 2$ by equating the corresponding values of $\bar{\theta}$ and $\bar{v}$. A specific procedure for accomplishing this may be described as follows:

(i) Choose an integer k_1 , $1 \leq k_1 < k$, so that the two groups of constants $a_1, a_2, \ldots, a_{k_1}$ and $a_{k_1+1}, a_{k_1+2}, \ldots, a_k$ are as homogeneous as possible.

(ii) Compute the group averages

$$\bar{a}_1 = \frac{1}{k_1} \sum_{i=1}^{k_1} a_i , \qquad \bar{a}_2 = \frac{1}{k-k_1} \sum_{i=k_1+1}^{k} a_i .$$

(iii) Determine the equivalent observations $\overline{X}_1$ and $\overline{X}_2$ for the $k = 2$ case by setting

$$\overline{a}_1\overline{X}_1 + \overline{a}_2\overline{X}_2 = \overline{\theta} \quad \text{and} \quad \overline{a}_1^2\overline{X}_1 + \overline{a}_2^2\overline{X}_2 = \overline{v} ,$$

where $\overline{\theta}$ and $\overline{v}$ are given by (17) and (18), thereby obtaining

$$\overline{X}_1 = \frac{\overline{a}_2\overline{\theta} - \overline{v}}{\overline{a}_1(\overline{a}_2-\overline{a}_1)} \quad \text{and} \quad \overline{X}_2 = \frac{\overline{v} - \overline{a}_1\overline{\theta}}{\overline{a}_1(\overline{a}_2-\overline{a}_1)} .$$

(iv) Determine the normalized constants a_1^* , a_2^* for the approximately equivalent $k = 2$ case by

$$a_1^* = \overline{a}_1/(\overline{a}_1+\overline{a}_2) , \qquad a_2^* = \overline{a}_2/(\overline{a}_1+\overline{a}_2) .$$

(v) Find the upper $1 - \alpha$ confidence bound θ^* for $k = 2$ using a_1^* , a_2^* , $\overline{X}_1$ and $\overline{X}_2$ and Table 3 or Table 4, interpolating as necessary.

(vi) Compute the $1 - \alpha$ upper confidence bound θ^{**} , say, for the original $(k \geq 3)$ problem by the formula

$$\theta^{**} = (\overline{a}_1+\overline{a}_2)\theta^* ,$$

where θ^* is the value found in (v) .

The validity of this procedure was checked for a number of examples involving $k = 3$ or $k = 4$ using the M.L.R. bound $\tilde{\theta}$ computed for the original example and the corresponding approximately equivalent $k = 2$ case (using $\overline{a}_1$ and $\overline{a}_2$ instead of a_1^* and a_2^* for comparability). The

agreement in the corresponding values of $\hat{\theta}$ was found to be extraordinarily close, usually to within at least three significant figures. The procedure was also found to be very insensitive to the choice of k_1. It should be noted that since the asymptotic bound depends on two quantities ($\bar{\theta}$ and $\bar{v}$) it is not reasonable to attempt to further reduce cases involving $k \geq 2$ to the $k = 1$ case.

The proposed procedure is illustrated in the following example:

Example. Suppose $k = 3$, $a_1 = 1/6$, $a_2 = 1/3$, $a_3 = 1/2$, the observations are $X_1 = 2$, $X_2 = 3$, $X_3 = 5$, and we seek an upper 90% confidence bound for θ. Then from (17) and (18) we have $\bar{\theta} = 3.8333$ and $\bar{v} = 1.6389$. Choosing $k_1 = 2$ we obtain $\bar{a}_1 = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{3}\right) = .25$ and $\bar{a}_2 = a_2 = .50$. From the formulas of (iii) we obtain $\bar{X}_1 = 4.4444$ and $\bar{X}_2 = 5.4444$, and from (iv), $a_1^* = 1/3$ and $a_2^* = 2/3$. Since $\bar{X}_1$ and $\bar{X}_2$ are not integers we must interpolate in Table 3 $(\alpha = .10)$. For $a_1 = 1/3$ the relevant entries from Table 3 are as follows:

I	X_1	X_2	θ^*
63	4	5	7.339
81	4	6	8.176
71	5	5	7.592
90	5	6	8.533

Four point linear interpolation yields

$$\theta^* = (.5556)^2(7.339) + (.5556)(.4444)(8.176 + 7.592) + (.4444)^2(8.533) = 7.844 .$$

Thus from (vi) the 90% upper confidence bound for θ is given by

$$\theta^{**} = (7.844)(.25 + .50) = 5.883 .$$

The values of the M.L.R. bound $\tilde{\theta}$ for the original $k = 3$ problem and the reduced $k = 2$ problem are respectively 5.736 and 5.737 . The very close agreement between these values (even though they both underestimate θ^{**}) strongly supports the validity of the suggested procedure.

Appendix 1

This appendix contains the proofs of the four general propositions of Section 2 and the verification that Assumption A is satisfied for the Poisson confidence bound problem.

Proof of Proposition 1: Consider $\underline{x},\underline{y} \in \mathcal{X}$ such that $\underline{x} \succcurlyeq \underline{y}$. Then for all $\underline{\lambda} \in \Lambda$,

$$P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \underline{x}\} \geq P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \underline{y}\} ,$$

and hence for all t

$$\sup_{\underline{\lambda}\varepsilon S(t)} P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \underline{x}\} \geq \sup_{\underline{\lambda}\varepsilon S(t)} P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \underline{y}\} .$$

Thus, recalling the definition of $\theta^*(\underline{x})$ and letting $S^* = S(\theta^*(\underline{y})) = \{\underline{\lambda}: \theta(\underline{\lambda}) = \theta^*(\underline{y})\}$, we have

$$\sup_{\underline{\lambda}\varepsilon S^*} P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \underline{x}\} \geq \alpha ,$$

so that $\theta^*(\underline{x}) \geq \theta^*(\underline{y})$ by the monotonicity part of Assumption A, which completes the proof.

Proof of Proposition 2: For fixed arbitrary $\underline{\lambda} \in \Lambda$, let $t = \theta(\underline{\lambda})$. Then if θ^{**} represents any inverse of θ^* **taking** values in the equivalence classes of $\mathcal{X}$ generated by the ordering $(\succcurlyeq)$, we have

$$\begin{aligned} P_{\underline{\lambda}}\{\theta(\underline{\lambda}) \geq \theta^*(\underline{X})\} &= P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \theta^{**}(t)\} \\ &\leq \sup_{\underline{\lambda}\varepsilon S(t)} P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \theta^{**}(t)\} \\ &\leq h(t) = \alpha , \end{aligned}$$

by the definition of θ^* . Hence

$$P_{\underline{\lambda}}\{\theta(\underline{\lambda}) \leq \theta^*(\underline{X})\} \geq P_{\underline{\lambda}}\{\theta(\underline{\lambda}) < \theta^*(\underline{X})\} \geq 1 - \alpha \ ,$$

and the proof is complete.

Proof of Proposition 3: Proceeding contrapositively we assume that there exists a point $\underline{x}' \in \mathcal{X}$ such that $\tilde{\theta}(\underline{x}') < \theta^*(\underline{x}')$. Let $S' = S(\tilde{\theta}(\underline{x}')) = \{\underline{\lambda}: \theta(\underline{\lambda}) = \tilde{\theta}(\underline{x}')\}$. Then by Assumption A and the definition of θ^* we have

$$\sup_{\underline{\lambda} \varepsilon S'} P_{\underline{\lambda}}\{\underline{X} \preccurlyeq \underline{x}'\} > \alpha \ ,$$

or since $\tilde{\theta}$ is monotone in $(\succcurlyeq)$,

$$\sup_{\underline{\lambda} \varepsilon S'} P_{\underline{\lambda}}\{\tilde{\theta}(\underline{X}) \leq \tilde{\theta}(\underline{x}')\} > \alpha \ .$$

Thus, there exists a $\underline{\lambda}' \varepsilon S'$ such that

$$P_{\underline{\lambda}'}\{\tilde{\theta}(\underline{X}) \leq \theta(\underline{\lambda}')\} > \alpha \ ,$$

which contradicts the assumption that $\tilde{\theta}$ is a $1-\alpha$ level confidence bound and completes the proof.

Proof of Proposition 4: For any $1-\alpha$ level confidence bound $\tilde{\theta}$ let $\overline{\theta}(\underline{x}) = \sup_{\underline{y} \preccurlyeq \underline{x}} \tilde{\theta}(\underline{y})$. Then $\overline{\theta}$ is monotone in the ordering $(\succcurlyeq)$ and if it is also a $1-\alpha$ level confidence bound, Proposition 3 guarantees that

$\overline{\theta}(\underline{x}) \geq \theta^*(\underline{x})$ for all $\underline{x} \in \mathcal{X}$ which is the desired result. Suppose contrapositively that $\overline{\theta}$ is not a $1-\alpha$ level confidence bound. Then there exists a $\underline{\lambda}^o \in \Lambda$ such that

$$P_{\underline{\lambda}^o}\{\overline{\theta}(\underline{X}) \leq \theta(\underline{\lambda}^o)\} > \alpha .$$

But $\tilde{\theta}(x) \leq \overline{\theta}(\underline{x})$ for all $\underline{x} \in \mathcal{X}$, so that

$$P_{\underline{\lambda}^o}\{\tilde{\theta}(\underline{X}) \leq \theta(\underline{\lambda}^0)\} > \alpha ,$$

which contradicts the assumption that $\tilde{\theta}$ is a $1-\alpha$ level confidence bound and completes the proof.

We demonstrate that Assumption A is satisfied for the Poisson confidence bound problem under consideration by means of the following two lemmas: For each t let $\underline{\lambda}_t$ be a value of $\underline{\lambda}$ for which the supremum of Assumption A is attained, i.e., for which (12) is satisfied. Let $\overline{\theta}(\underline{x}) = \sum_{i=1}^{k} a_i X_i$, and let $\hat{\theta}(\underline{x})$ be as defined in (14).

<u>Lemma 1</u>. For $t > 0$, $\frac{\partial}{\partial t} P_{\underline{\lambda}_t}\{\underline{X} \lesssim \underline{x}\} < 0$ if and only if

$$E_{\underline{\lambda}_t}\{\overline{\theta}(\underline{X}) | \hat{\theta}(\underline{X}) \leq \hat{\theta}(\underline{x})\} < t .$$

<u>Proof</u>: Suppressing the dependence on t in the notation for the components of $\underline{\lambda}_t = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and recalling that $\sum_{i=1}^{k} a_i \lambda_i = t$ we see that $\frac{\partial \lambda_i}{\partial t} = 1/a_i$, $i = 1,2,\ldots,k$, which when combined with (11) yields

$$\text{(A1)} \qquad \frac{\partial}{\partial t} P_{\underline{\lambda}_t}\{\underline{X} \lesssim \underline{x}\} = \sum_{\underline{y} \lesssim \underline{x}} \sum_{j=1}^{k} \frac{1}{a_j}\left(\frac{y_j}{\lambda_j} - 1\right) P_{\underline{\lambda}_t}\{\underline{X} = \underline{y}\} .$$

Now from (12), summing on j yields

$$\text{(A2)} \qquad \sum_{\underline{y} \precneq \underline{x}} \sum_{j=1}^{k-1} \frac{a_k}{a_j^2} \left(\frac{y_k}{\lambda_k} - 1 \right) P_{\underline{\lambda}_t} \{\underline{X} = \underline{y}\} = \sum_{\underline{y} \leq x} \sum_{j=1}^{k=1} \frac{1}{a_j} \left(\frac{y_j}{\lambda_j} - 1 \right) P_{\underline{\lambda}_t} \{\underline{X} = \underline{y}\} .$$

From (A1) and (A2) we obtain

$$\text{(A3)} \qquad \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{\underline{X} \precsim \underline{x}\} = \left(\sum_{j=1}^{k} \frac{1}{a_j^2} \right) \sum_{\underline{y} \precneq \underline{x}} a_k \left(\frac{y_k}{\lambda_k} - 1 \right) P_{\underline{\lambda}_t} \{\underline{X} = \underline{y}\} .$$

By symmetry (A3) must also hold with a_i, y_i, λ_i replacing a_k, y_k, λ_k respectively for $i = 1,2,\ldots,k-1$. Hence multiplying each of these equations by the corresponding λ_i and summing we obtain

$$\text{(A4)} \qquad \left(\sum_{i=1}^{k} \lambda_i \right) \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{\underline{X} \precsim \underline{x}\} = \left(\sum_{i=1}^{k} \frac{1}{a_i^2} \right) \sum_{\underline{y} \precneq \underline{x}} (\overline{\theta}(\underline{y}) - t) \; P_{\underline{\lambda}_t} \{\underline{X} = \underline{y}\} ,$$

which yields the desired result upon observing that

$$E_{\underline{\lambda}_t} \{\overline{\theta}(\underline{X}) \,|\, \underline{X} \precsim \underline{y}\} = \left(P_{\underline{\lambda}_t} \{\underline{X} \precsim \underline{x}\} \right)^{-1} \sum_{\underline{y} \precsim \underline{x}} \overline{\theta}(\underline{y}) \; P_{\underline{\lambda}_t} \{\underline{X} = \underline{y}\} ,$$

and recalling that $\underline{x} \precsim \underline{y} \iff \hat{\theta}(\underline{x}) \leq \hat{\theta}(\underline{y})$. Note that since $t > 0$, $\underline{\lambda}_t \neq 0$ and hence $\sum_{i=1}^{k} \lambda_i > 0$.

Now consider a random vector $\underline{Z} = (Z_1, Z_2, \ldots, Z_n)$ where the Z_i's are independent non-negative random variables. Let C be the hypercube in k-dimensional space defined for $c \in (0,\infty)$ by

$$C = \left\{ \underline{z} = (z_1, z_2, \ldots, z_k) : \; 0 \leq z_i \leq c \;, \quad i = 1,2,\ldots,k \right\} .$$

For any measurable set R in the non-negative orthant of k-space, for $v \geq 0$ let

$$R_i(v) = R \cap \{\underline{z}: z_i = v\}$$

for $i = 1,2,\ldots,k$.

Lemma 2.[1] If $R \subset C$, $P\{\underline{Z} \varepsilon R\} > 0$, and if $v > v'$ implies $R_i(v) \subset R_i(v')$ for $i = 1,2,\ldots,k$, then

$$E\{Z_i | \underline{Z} \varepsilon R\} \leq E\{Z_i | Z_i \leq c\}$$

for $i = 1,2,\ldots,k$.

Proof: Let $f_i(z)$, $i = 1,2,\ldots,k$, be the probability densities of the Z_i's with respect to a dominating measure $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_k$. Then taking $i = 1$ and letting $\mu^{(k-1)} = \mu_2 \times \cdots \times \mu_k$, $\underline{u} = (u_2,u_3,\ldots,u_k)$, $P_1(c) = P\{Z_1 \leq c\}$, and $f^{(k-1)}(\underline{u}) = \prod_{i=2}^{k} f_i(u_i)$, we have

$$\text{(A5)} \qquad E\{Z_1 | \underline{Z} \varepsilon R\} = \frac{1}{P\{\underline{Z} \varepsilon R\}} \int_0^c v\, f_1(v) \int_{R_1(v)} f^{(k-1)}(\underline{u})\, d\mu^{(k-1)}\, d\mu_1 .$$

But the inner integral on the right is descreasing in v so that by the well known inequality which states that if g_1 and g_2 are two functions monotone in opposite senses and Y is a random variable, then $E\, g_1(Y)\, g_2(Y) \leq E\, g_1(Y)\, E\, g_2(Y)$, we have

[1]The author benefited from helpful discussions with Rupert G. Miller concerning the proof of this lemma.

$$\int_0^c \left(v \int_{R_1(v)} f^{(k-1)}(\underline{u})\, d\mu^{(k-1)} \right) \frac{f_1(v)}{P_1(c)}\, d\mu_1$$

$$\text{(A6)} \qquad \leq \int_0^c v\, \frac{f_1(v)}{P_1(c)}\, d\mu_1 \quad \int_0^c \left(\int_{R_1(v)} f^{(k-1)}(\underline{u})\, d\mu^{(k-1)} \right) \frac{f_1(v)}{P_1(c)}\, d\mu_1$$

$$= E\{Z_1 | Z_1 \leq c\}\; P\{\underline{Z} \in R\} \big/ P_1(c) \; .$$

Hence by (A5) and (A6)

$$E\{Z_1 | \underline{Z} \in R\} \leq E\{Z_1 | Z_1 \leq c\} \; ,$$

and the proof is complete.

We observe that for any i, $E\{Z_i | Z_i \leq c\} < E\, Z_i$, provided that $P\{Z_i \leq c\} < 1$, which implies that the conditional c.d.f. of Z_i, given $\{Z_i \leq c\}$, is strictly greater than the c.d.f. of Z_i on the interval $(0,c)$.

To verify that Assumption A is satisfied for the Poisson model under consideration, let $Z_i = a_i X_i$, $i = 1,2,\ldots,k$, so that $\hat{\theta}$ given by (14) becomes

$$\hat{\theta}(\underline{X}) = \sum_{i=1}^{k} Z_i + c_\alpha \left(\sum_{i=1}^{n} a_i Z_i \right)^{1/2} \stackrel{\text{def.}}{=} g(\underline{Z}) \; .$$

Let $\underline{z} = (z_1, z_2, \ldots, z_k)$ and for fixed arbitrary $\underline{x} \in \mathfrak{X}$, let

$$R = \{\underline{z} : \; \underline{z} \geq 0 \; , \; g(\underline{z}) \leq \hat{\theta}(\underline{x})\} \; .$$

Since $\frac{\partial}{\partial z_i} g(\underline{z}) > 0$ for all $\underline{z} \geq 0$, we conclude that for $v > v'$, $\underline{z} \in R_i(v)$ implies $\underline{z} \in R_i(v')$ for $i = 1,2,\ldots,k$. Also, for sufficiently

large $c > 0$, $R \subset C$ and $P_{\underline{\lambda}}\{\underline{Z} \varepsilon R\} > 0$, so that the conditions of Lemma 2 are satisfied. Also, if $\lambda_i > 0$, then $P\{Z_i \leq c\} < 1$ and by the remark following Lemma 2, $E\{Z_i | Z_i \leq c\} < E\, Z_i$. Hence noting that $\overline{\theta}(\underline{X}) = \sum_{i=1}^{k} Z_i$ we have for $\underline{\lambda} \neq 0$,

$$E_{\underline{\lambda}}\{\overline{\theta}(\underline{X}) | \hat{\theta}(\underline{X}) \leq \hat{\theta}(\underline{x})\} = \sum_{i=1}^{k} E_{\underline{\lambda}}\{Z_i | \underline{Z} \varepsilon R\}$$

$$< \sum_{i=1}^{k} E_{\underline{\lambda}}\, Z_i = E_{\underline{\lambda}}\, \overline{\theta}(\underline{X}) .$$

Thus, recalling that $E_{\underline{\lambda}_t} \overline{\theta}(\underline{X}) = \sum_{i=1}^{k} a_i \lambda_i = t$, we have by Lemma 1,

$$\frac{\partial}{\partial t} h_{\underline{x}}(t) = \frac{\partial}{\partial t} P_{\underline{\lambda}_t} \{\underline{X} \leq \underline{x}\} < 0 ,$$

and therefore $h_{\underline{x}}(t)$ is decreasing in t for all $x \varepsilon \mathfrak{X}$. The existence of a smallest value of $t_\alpha(\underline{x})$ follows from the continuity and form of the left-hand members of equations (12) and (13) and Assumption A is therefore satisfied.

Appendix 2

This appendix contains the detailed development of the approximations employed in Section 3. These approximations are based on the asymptotic properties of the maximum likelihood ratio (M.L.R.) test statistic.

As before, let $\underline{X} = (X_1, X_2, \ldots, X_k)$ and $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where for each i, X_i is a Poisson random variable with parameter λ_i and the X_i's are independent. To obtain the M.L.R. test of H_0: $\theta = \theta_0$ vs. H_1: $\theta \neq \theta_0$, we let $\Lambda_0 = \{\underline{\lambda}: \theta_0 = \sum_{i=1}^{k} a_i \lambda_i\}$ and observe that the likelihood function is

$$L(\underline{\lambda}, \underline{X}) = \exp\{\sum_{i=1}^{k} (X_i \log \lambda_i - \lambda_i - \log X_i!)\} .$$

Since the unconstrained maximum likelihood estimate (M.L.E.) of λ_i is X_i, $i = 1,2,\ldots,k$, we have

$$\max_{\Lambda} \log L(\underline{\lambda}, \underline{X}) = \sum_{i=1}^{k} (X_i \log X_i - X_i - \log X_i!) .$$

To obtain the M.L.E.'s under the constraint that $\underline{\lambda} \varepsilon \Lambda_0$ (i.e., H_0 is true) we use the Lagrange multiplier method and observe that for $j = 1,2,\ldots,k$

$$\frac{\partial}{\partial \lambda_j} \{\log L(\underline{X}, \underline{\lambda}) + \mu(\sum_{i=1}^{k} a_i \lambda_i - \theta_0)\} = 0$$

if and only if

$$\frac{X_j}{\lambda_j} + a_j \mu = 1 .$$

Thus the restricted M.L.E.'s are

$$\hat{\lambda}_i = X_i/(1-a_i\hat{\mu}) ,$$

for $i = 1,2,\ldots,k$, with $\hat{\mu} = \hat{\mu}(\theta_0)$ determined by

$$\sum_{i=1}^{k} a_i X_i/(1-a_i\hat{\mu}) = \theta_0 . \tag{A7}$$

Thus, if $S = \max_{\Lambda_0} L/\max_{\Lambda} L$, the M.L.R. statistic is

$$\begin{aligned} T &= -2 \log S \\ &= 2 \sum_{i=1}^{k} \left\{X_i \log X_i - X_i - X_i \log\left(\frac{X_i}{1-a_i\hat{\mu}}\right) + \frac{X_i}{1-a_i\hat{\mu}}\right\} \\ &= 2 \sum_{i-1}^{k} X_i \left\{\frac{a_i\hat{\mu}}{1-a_i\hat{\mu}} + \log(1-a_i\hat{\mu})\right\} , \end{aligned} \tag{A8}$$

or

$$T = 2\theta_0\hat{\mu} + 2 \sum_{i=1}^{k} X_i \log(1-a_i\hat{\mu}) . \tag{A9}$$

If T were based on n i.i.d. observations we would expect T to be asymptotically chi-squared with one degree of freedom as $n \to \infty$. For the present case (since k is fixed) a slightly different analysis is needed. Let $Z_i = \lambda_i^{-1/2}(X_i-\lambda_i)$, $i = 1,2,\ldots,k$. Then the Z_i's are bounded in probability and if $\lambda_i \to \infty$, then Z_i is asymptotically a standard normal variable. Now $X_i = \lambda_i^{1/2} Z_i + \lambda_i$ and from (A7)

$$\sum_{i=1}^{k} \frac{a_i \lambda_i^{1/2} Z_i + a_i\lambda_i}{1 - a_i\hat{\mu}} = \theta_0 = \sum_{i=1}^{k} a_i\lambda_i ,$$

and hence

$$\sum_{i=1}^{k} \frac{a_i(\lambda_i^{1/2} Z_i + a_i\lambda_i\hat{\mu})}{1 - a_i\hat{\mu}} = 0 . \tag{A10}$$

Let $\gamma = \max(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and consider γ large. Then using the symbol $0_p(g(\gamma))$ to represent a random quantity which when divided by $g(\gamma)$ is bounded in probability as $\gamma \to \infty$, we observe that (A10) implies $\hat{\mu} = 0_p(\gamma^{-1/2})$ so that $(1 - a_i\hat{\mu})^{-1} = 1 + 0_p(\gamma^{-1/2})$ for each i, and hence

$$\text{(A11)} \qquad \hat{\mu} = -\frac{\sum_{i=1}^{k} a_i \lambda_i^{1/2} Z_i}{\sum_{i=1}^{k} a_i^2 \lambda_i} (1 + 0_p(\gamma^{-1/2})) .$$

Now $\log(1 - a_i\hat{\mu}) = -a_i\hat{\mu} - \frac{1}{2} a_i^2\hat{\mu}^2 + 0_p(\gamma^{-3/2})$, so that (A9) becomes

$$T = 2 \left\{ \hat{\mu} \sum_{i=1}^{k} a_i\lambda_i - \sum_{i=1}^{k} X_i \left(a_i\hat{\mu} + \frac{1}{2} a_i^2\hat{\mu}^2 + 0_p(\gamma^{-3/2}) \right) \right\}$$

$$= 2 \left\{ -\hat{\mu} \sum_{i=1}^{k} a_i \lambda_i^{1/2} Z_i - \frac{1}{2} \hat{\mu}^2 \sum_{i=1}^{k} a_i^2\lambda_i \right\} + 0_p(\gamma^{-1/2}) .$$

Recalling (A11) we have

$$T = 2 \left\{ \hat{\mu}^2 (1 + 0_p(\gamma^{-1/2})) \sum_{i=1}^{k} a_i^2\lambda_i - \frac{1}{2} \hat{\mu}^2 \sum_{i=1}^{k} a_i^2\lambda_i \right\} + 0_p(\gamma^{-1/2})$$

$$\text{(A12)} \qquad = \hat{\mu}^2 \sum_{i=1}^{k} a_i^2\lambda_i + 0_p(\gamma^{-1/2}) .$$

Hence, letting ε_γ represent a generic random quantity approaching zero in probability as $\gamma \to \infty$, we have by (A11) and (A12),

$$T = \left[\sum_{i=1}^{k} a_i \lambda_i^{1/2} Z_i \Big/ \left(\sum_{i=1}^{k} a_i^2 \lambda_i \right)^{1/2} \right]^2 + \varepsilon_\gamma$$

$$\text{(A13)} \qquad = \left[\left(\sum_{i=1}^{k} a_i X_i - \theta_0 \right) \Big/ \left(\sum_{i=1}^{k} a_i \lambda_i \right)^{1/2} \right]^2 + \varepsilon_\gamma$$

$$= \left[\left(\sum_{i=1}^{k} a_i X_i - \theta_0 \right) \Big/ \left(\sum_{i=1}^{k} a_i X_i \right)^{1/2} \right]^2 + \varepsilon_\gamma ,$$

since

$$\left(\sum_{i=1}^{k} a_i^2 X_i \right)^{1/2} = \left(\sum_{i=1}^{k} a_i^2 \lambda_i \right)^{1/2} (1 + \varepsilon_\gamma) .$$

Thus, for large γ, T is approximately the square of a standard normal variate and the approximate two-sided test of H_0: $\theta=\theta_0$ at the α significance level is to reject if $T > \chi^2_{1,\alpha}$, where $\chi^2_{1,\alpha}$ is the upper α^{th} quantile of the chi-squared distribution with one degree of freedom. The corresponding asymptotic two-sided confidence statement for θ is

$$\sum_{i=1}^{k} a_i X_i - c_{\alpha/2} \left(\sum_{i=1}^{k} a_i^2 X_i \right)^{1/2} \leq \theta \leq \sum_{i=1}^{k} a_i X_i + c_{\alpha/2} \left(\sum_{i=1}^{k} a_i^2 X_i \right)^{1/2}$$

with probability $1 - \alpha$, where $c_{\alpha/2}$ is the upper $(\frac{1}{2}\alpha)^{th}$ quantile of the standard normal distribution. The corresponding one-sided confidence statement at level $1 - \alpha$ is $\theta \leq \hat{\theta}$ where $\hat{\theta}$ is given by (14). To obtain an asymptotic upper (one-sided) level $1 - \alpha$ confidence bound for θ using the M.L.R. statistic directly, one first finds the root $\hat{\mu}_0$ lying in the interval $(0, 1/\max(a_1, a_2, \ldots, a_k))$ of the equation

(A14) $$T = \chi^2_{1,2\alpha} ,$$

where T is given by (A8). Substituting $\hat{\mu}_0$ for $\hat{\mu}$ in (A7) then yields $\theta_0 = \tilde{\theta}$, the required upper $1 - \alpha$ confidence bound for θ . The negative root of (A14) will yield a lower confidence bound for θ .

Appendix 3

Tables 3 and 4 for the case $k = 2$ were produced by solving (12) and (13) simultaneously for λ_1^* and λ_2^* by numerical methods and computing θ^* from (10). The numerical methods used involved a finitized analogue of Newton's method for two dimensions. The calculations were done in part on a self-contained mini-computer system (Wang 2200) and in part on an IBM 360/60 system. In a small proportion of the cases considered the procedure failed to converge in a reasonable number of iterations and the tabulated values were obtained by interpolation according to the values of $\hat{\theta}$ given by (14) subject to the requirement that θ^* be monotone in the given ordering.

An examination of Tables 3 and 4 shows that often the same or very nearly the same value of θ^* occurs for several successive sample points. This reflects the fact that at certain stages an additional pair $(x_1,x_2) = \underline{x}$ may contribute very little to the quantity $P_{\underline{\lambda}}\{\underline{X} \lesssim \underline{x}\}$ in the vicinity of the solution pair $(\lambda_1^*,\lambda_2^*) = \underline{\lambda}^*$. In higher dimensions $(k > 2)$ this phenomenon might be exploited to condense the corresponding tables considerably.

Tables 1 and 2 were computed by directly generating the required Poisson probabilities rather than by the traditional method utilizing the corresponding chi-squared probabilities.

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TABLE 1

ALPHA=.10 K=1

X	THETA*	X	THETA*	X	THETA*
0	2.303	50	60.339	100	114.075
1	3.890	51	61.429	101	115.138
2	5.322	52	62.518	102	116.202
3	6.681	53	63.605	103	117.265
4	7.994	54	64.692	104	118.327
5	9.275	55	65.779	105	119.390
6	10.532	56	66.864	106	120.452
7	11.771	57	67.949	107	121.514
8	12.995	58	69.033	108	122.576
9	14.206	59	70.116	109	123.637
10	15.407	60	71.199	110	124.698
11	16.598	61	72.281	111	125.759
12	17.782	62	73.362	112	126.819
13	18.958	63	74.442	113	127.879
14	20.128	64	75.523	114	128.939
15	21.292	65	76.602	115	129.999
16	22.452	66	77.680	116	131.059
17	23.606	67	78.759	117	132.118
18	24.756	68	79.836	118	133.177
19	25.902	69	80.913	119	134.235
20	27.045	70	81.990	120	135.294
21	28.184	71	83.066	121	136.352
22	29.320	72	84.141	122	137.410
23	30.453	73	85.216	123	138.468
24	31.583	74	86.291	124	139.525
25	32.711	75	87.364	125	140.582
26	33.836	76	88.438	126	141.640
27	34.959	77	89.511	127	142.696
28	36.080	78	90.583	128	143.753
29	37.198	79	91.655	129	144.809
30	38.315	80	92.727	130	145.865
31	39.430	81	93.798	131	146.921
32	40.543	82	94.869	132	147.977
33	41.654	83	95.939	133	149.033
34	42.764	84	97.009	134	150.088
35	43.872	85	98.078	135	151.143
36	44.978	86	99.147	136	152.198
37	46.083	87	100.216	137	153.253
38	47.187	88	101.284	138	154.307
39	48.289	89	102.352	139	155.361
40	49.390	90	103.419	140	156.416
41	50.490	91	104.486	141	157.470
42	51.589	92	105.553	142	158.523
43	52.686	93	106.620	143	159.577
44	53.782	94	107.686	144	160.630
45	54.878	95	108.751	145	161.683
46	55.972	96	109.817	146	162.736
47	57.065	97	110.882	147	163.789
48	58.158	98	111.946	148	164.842
49	59.249	99	113.011	149	165.894

TABLE 2

ALPHA=.05 K=1

X	THETA*	X	THETA*	X	THETA*
0	2.996	50	63.287	100	118.078
1	4.744	51	64.402	101	119.160
2	6.296	52	65.516	102	120.241
3	7.754	53	66.628	103	121.322
4	9.153	54	67.740	104	122.403
5	10.513	55	68.850	105	123.484
6	11.842	56	69.960	106	124.564
7	13.148	57	71.069	107	125.643
8	14.434	58	72.176	108	126.722
9	15.705	59	73.283	109	127.801
10	16.962	60	74.389	110	128.879
11	18.207	61	75.494	111	129.957
12	19.443	62	76.599	112	131.035
13	20.669	63	77.702	113	132.112
14	21.886	64	78.805	114	133.189
15	23.097	65	79.906	115	134.265
16	24.301	66	81.007	116	135.342
17	25.499	67	82.108	117	136.418
18	26.692	68	83.207	118	137.493
19	27.879	69	84.306	119	138.568
20	29.062	70	85.404	120	139.643
21	30.240	71	86.501	121	140.718
22	31.415	72	87.598	122	141.792
23	32.585	73	88.694	123	142.866
24	33.752	74	89.790	124	143.940
25	34.916	75	90.884	125	145.014
26	36.076	76	91.979	126	146.087
27	37.234	77	93.072	127	147.160
28	38.389	78	94.165	128	148.232
29	39.541	79	95.257	129	149.305
30	40.690	80	96.349	130	150.376
31	41.837	81	97.441	131	151.449
32	42.982	82	98.531	132	152.520
33	44.125	83	99.621	133	153.592
34	45.266	84	100.711	134	154.663
35	46.404	85	101.800	135	155.733
36	47.541	86	102.888	136	156.804
37	48.675	87	103.977	137	157.873
38	49.808	88	105.064	138	158.944
39	50.940	89	106.151	139	160.014
40	52.069	90	107.238	140	161.083
41	53.197	91	108.324	141	162.152
42	54.324	92	109.409	142	163.221
43	55.449	93	110.494	143	164.290
44	56.572	94	111.579	144	165.359
45	57.695	95	112.663	145	166.427
46	58.816	96	113.747	146	167.495
47	59.935	97	114.831	147	168.563
48	61.054	98	115.914	148	169.630
49	62.171	99	116.996	149	170.698

TABLE 3

ALPHA= .10 A1=.10 K=2

I	X1	X2	THETA*
1	0	0	2.072
2	1	0	2.079
3	2	0	2.111
4	3	0	2.163
5	4	0	2.225
6	5	0	2.293
7	6	0	2.367
8	7	0	2.444
9	8	0	2.524
10	9	0	2.606
11	10	0	2.690
12	11	0	2.776
13	12	0	2.863
14	13	0	2.951
15	14	0	3.040
16	15	0	3.130
17	0	1	3.501
18	16	0	3.501
19	1	1	3.507
20	17	0	3.507
21	2	1	3.540
22	18	0	3.540
23	3	1	3.592
24	19	0	3.592
25	4	1	3.654
26	20	0	3.654
27	5	1	3.723
28	21	0	3.723
29	6	1	3.797
30	22	0	3.797
31	7	1	3.875
32	8	1	3.955
33	23	0	3.955
34	9	1	4.038
35	24	0	4.038
36	10	1	4.123
37	25	0	4.123
38	11	1	4.209
39	26	0	4.209
40	12	1	4.297
41	27	0	4.297
42	0	2	4.790
43	13	1	4.790
44	28	0	4.790
45	1	2	4.796
46	14	1	4.796
47	29	0	4.796
48	2	2	4.829
49	15	1	4.829
50	30	0	4.829

I	X1	X2	THETA*
51	3	2	4.881
52	16	1	4.881
53	31	0	4.881
54	4	2	4.943
55	17	1	4.943
56	32	0	4.943
57	5	2	5.013
58	18	1	5.013
59	33	0	5.013
60	6	2	5.087
61	19	1	5.087
62	34	0	5.087
63	7	2	5.165
64	20	1	5.165
65	35	0	5.165
66	8	2	5.246
67	21	1	5.246
68	36	0	5.246
69	9	2	5.329
70	22	1	5.329
71	37	0	5.329
72	10	2	5.414
73	23	1	5.414
74	11	2	5.501
75	38	0	5.501
76	24	1	5.501
77	12	2	5.589
78	0	3	6.013
79	39	0	6.013
80	25	1	6.013
81	13	2	6.013
82	1	3	6.019
83	40	0	6.019
84	26	1	6.019
85	14	2	6.019
86	2	3	6.052
87	41	0	6.052
88	27	1	6.052
89	15	2	6.052
90	3	3	6.104
91	42	0	6.104
92	28	1	6.104
93	16	2	6.104
94	4	3	6.166
95	43	0	6.166
96	29	1	6.166
97	17	2	6.166
98	5	3	6.236
99	44	0	6.236
100	30	1	6.236

TABLE 3 (CONT.)

ALPHA= .10 A1=.20 K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	1.842	51	20	0	5.810
2	1	0	1.872	52	15	1	5.810
3	2	0	1.979	53	0	4	6.395
4	3	0	2.123	54	10	2	6.395
5	4	0	2.286	55	5	3	6.395
6	5	0	2.460	56	21	0	6.395
7	0	1	3.111	57	16	1	6.395
8	6	0	3.111	58	1	4	6.414
9	1	1	3.142	59	11	2	6.431
10	7	0	3.142	60	6	3	6.434
11	2	1	3.251	61	22	0	6.536
12	8	0	3.251	62	17	1	6.536
13	3	1	3.398	63	2	4	6.536
14	9	0	3.399	64	12	2	6.536
15	4	1	3.567	65	7	3	6.536
16	10	0	3.567	66	23	0	6.536
17	5	1	3.748	67	18	1	6.536
18	11	0	3.751	68	3	4	6.686
19	0	2	4.258	69	8	3	6.688
20	6	1	4.258	70	13	2	6.688
21	1	2	4.288	71	24	0	6.688
22	12	0	4.288	72	4	4	6.857
23	7	1	4.288	73	19	1	6.857
24	2	2	4.398	74	9	3	6.864
25	13	0	4.398	75	14	2	6.867
26	8	1	4.398	76	25	0	7.027
27	3	2	4.548	77	0	5	7.420
28	14	0	4.548	78	5	4	7.420
29	9	1	4.548	79	20	1	7.420
30	4	2	4.719	80	10	3	7.420
31	15	0	4.719	81	15	2	7.420
32	10	1	4.721	82	26	0	7.420
33	5	2	4.904	83	1	5	7.420
34	0	3	5.344	84	6	4	7.463
35	16	0	5.344	85	21	1	7.479
36	11	1	5.344	86	11	3	7.490
37	6	2	5.344	87	16	2	7.498
38	1	3	5.362	88	2	5	7.561
39	17	0	5.438	89	27	0	7.561
40	12	1	5.486	90	7	4	7.562
41	7	2	5.486	91	22	1	7.562
42	2	3	5.486	92	12	3	7.562
43	18	0	5.486	93	17	2	7.562
44	13	1	5.486	94	3	5	7.712
45	8	2	5.486	95	28	0	7.712
46	3	3	5.636	96	8	4	7.714
47	19	0	5.636	97	23	1	7.714
48	14	1	5.636	98	13	3	7.714
49	9	2	5.637	99	18	2	7.714
50	4	3	5.810	100	4	5	7.884

TABLE 3 (CONT.)

ALPHA= .10 A1=.25 K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	1.727	51	17	0	6.224
2	1	0	1.781	52	6	3	6.230
3	2	0	1.944	53	10	2	6.230
4	3	0	2.151	54	3	4	6.443
5	4	0	2.381	55	14	1	6.443
6	0	1	2.917	56	18	0	6.443
7	5	0	2.917	57	7	3	6.498
8	1	1	2.972	58	0	5	6.956
9	6	0	2.972	59	11	2	6.956
10	2	1	3.142	60	4	4	6.956
11	7	0	3.142	61	15	1	6.956
12	3	1	3.362	62	8	3	6.956
13	0	2	3.991	63	19	0	6.956
14	4	1	3.991	64	1	5	6.986
15	8	0	3.991	65	12	2	7.028
16	1	2	4.047	66	5	4	7.028
17	5	1	4.047	67	16	1	7.028
18	9	0	4.047	68	9	3	7.028
19	2	2	4.217	69	20	0	7.028
20	6	1	4.220	70	2	5	7.044
21	10	0	4.220	71	13	2	7.139
22	3	2	4.433	72	6	4	7.215
23	7	1	4.450	73	17	1	7.215
24	11	0	4.450	74	10	3	7.286
25	0	3	5.011	75	21	0	7.294
26	4	2	5.011	76	3	5	7.407
27	8	1	5.011	77	14	2	7.407
28	12	0	5.011	78	7	4	7.449
29	1	3	5.066	79	0	6	7.899
30	5	2	5.066	80	18	1	7.899
31	9	1	5.066	81	11	3	7.899
32	13	0	5.066	82	4	5	7.899
33	2	3	5.238	83	22	0	7.899
34	6	2	5.242	84	15	2	7.899
35	10	1	5.242	85	8	4	7.899
36	14	0	5.242	86	1	6	7.900
37	3	3	5.456	87	19	1	7.914
38	7	2	5.489	88	12	3	7.914
39	0	4	5.995	89	5	5	7.914
40	11	1	5.995	90	23	0	7.914
41	15	0	5.995	91	16	2	7.914
42	4	3	5.995	92	2	6	8.130
43	8	2	5.995	93	9	4	8.130
44	1	4	6.051	94	20	1	8.130
45	12	1	6.051	95	13	3	8.130
46	16	0	6.051	96	6	5	8.136
47	5	3	6.051	97	24	0	8.136
48	9	2	6.051	98	17	2	8.136
49	2	4	6.224	99	3	6	8.351
50	13	1	6.224	100	10	4	8.355

TABLE 3 (CONT.)

ALPHA= .10 A1=.30 K=2

I	X1	X2	THETA*
1	0	0	1.612
2	1	0	1.703
3	2	0	1.941
4	3	0	2.229
5	0	1	2.723
6	4	0	2.723
7	1	1	2.819
8	2	1	3.054
9	5	0	3.074
10	0	2	3.726
11	3	1	3.726
12	6	0	3.726
13	1	2	3.820
14	4	1	3.823
15	7	0	3.823
16	2	2	4.059
17	5	1	4.094
18	8	0	4.096
19	0	3	4.676
20	3	2	4.676
21	6	1	4.676
22	9	0	4.676
23	1	3	4.772
24	4	2	4.776
25	7	1	4.776
26	10	0	4.776
27	2	3	5.012
28	5	2	5.060
29	8	1	5.087
30	11	0	5.101
31	0	4	5.597
32	3	3	5.597
33	6	2	5.597
34	9	1	5.597
35	12	0	5.597
36	1	4	5.691
37	4	3	5.696
38	7	2	5.696
39	10	1	5.696
40	2	4	5.932
41	13	0	5.932
42	5	3	5.991
43	8	2	6.040
44	0	5	6.492
45	11	1	6.492
46	3	4	6.492
47	14	0	6.492
48	6	3	6.492
49	9	2	6.492
50	1	5	6.589
51	12	1	6.589
52	4	4	6.594
53	15	0	6.594
54	7	3	6.594
55	10	2	6.594
56	2	5	6.830
57	13	1	6.830
58	5	4	6.898
59	16	0	6.898
60	8	3	6.962
61	0	6	7.372
62	11	2	7.372
63	3	5	7.372
64	14	1	7.372
65	6	4	7.372
66	17	0	7.372
67	1	6	7.469
68	9	3	7.469
69	12	2	7.469
70	4	5	7.475
71	15	1	7.475
72	7	4	7.475
73	18	0	7.475
74	2	6	7.711
75	10	3	7.711
76	13	2	7.711
77	5	5	7.787
78	0	7	8.239
79	16	1	8.239
80	8	4	8.239
81	19	0	8.239
82	3	6	8.239
83	11	3	8.239
84	6	5	8.239
85	14	2	8.239
86	1	7	8.337
87	17	1	8.337
88	9	4	8.337
89	20	0	8.337
90	4	6	8.344
91	12	3	8.344
92	7	5	8.455
93	15	2	8.462
94	2	7	8.579
95	18	1	8.579
96	10	4	8.579
97	5	6	8.661
98	21	0	8.661
99	13	3	8.661
100	0	8	9.096

TABLE 3 (CONT.)

ALPHA= .10 A1=1/3 K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	1.534	51	0	6	6.668
2	1	0	1.663	52	9	2	6.668
3	2	0	1.965	53	7	3	6.668
4	0	1	2.593	54	14	0	6.720
5	3	0	2.593	55	5	4	6.755
6	1	1	2.727	56	12	1	6.755
7	4	0	2.734	57	3	5	6.984
8	2	1	3.031	58	1	6	7.164
9	0	2	3.548	59	10	2	7.164
10	5	0	3.548	60	8	3	7.164
11	3	1	3.548	61	15	0	7.164
12	1	2	3.685	62	6	4	7.164
13	6	0	3.685	63	4	5	7.339
14	4	1	3.775	64	13	1	7.339
15	2	2	3.991	65	2	6	7.474
16	7	0	3.994	66	11	2	7.474
17	0	3	4.454	67	0	7	7.515
18	5	1	4.454	68	9	3	7.515
19	3	2	4.454	69	16	0	7.515
20	1	3	4.593	70	7	4	7.515
21	8	0	4.593	71	5	5	7.592
22	6	1	4.593	72	14	1	7.592
23	4	2	4.718	73	3	6	7.671
24	2	3	4.900	74	12	2	7.702
25	9	0	4.901	75	1	7	7.990
26	0	4	5.329	76	10	3	7.990
27	7	1	5.329	77	8	4	7.990
28	5	2	5.329	78	17	0	7.990
29	3	3	5.329	79	6	5	7.991
30	10	0	5.329	80	15	1	7.991
31	1	4	5.469	81	4	6	8.176
32	8	1	5.469	82	13	2	8.176
33	6	2	5.470	83	2	7	8.302
34	4	3	5.592	84	11	3	8.302
35	11	0	5.618	85	0	8	8.310
36	2	4	5.778	86	9	4	8.310
37	9	1	5.779	87	18	0	8.320
38	0	5	5.820	88	7	5	8.347
39	7	2	5.834	89	16	1	8.422
40	5	3	5.902	90	5	6	8.533
41	12	0	5.902	91	14	2	8.533
42	3	4	6.130	92	3	7	8.648
43	10	1	6.131	93	12	3	8.648
44	1	5	6.324	94	1	8	8.807
45	8	2	6.324	95	10	4	8.807
46	6	3	6.325	96	19	0	8.807
47	13	0	6.351	97	8	5	8.807
48	4	4	6.488	98	17	1	8.807
49	11	1	6.490	99	6	6	8.808
50	2	5	6.634	100	15	2	8.808

TABLE 3 (CONT.)

ALPHA= .10 A1=.40 K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	1.381	51	5	4	6.767
2	1	0	1.640	52	10	1	6.814
3	0	1	2.334	53	2	6	6.986
4	2	0	2.334	54	7	3	6.986
5	1	1	2.563	55	12	0	6.986
6	3	0	2.681	56	4	5	6.987
7	0	2	2.938	57	9	2	7.134
8	2	1	3.079	58	1	7	7.134
9	4	0	3.079	59	6	4	7.278
10	1	2	3.416	60	11	1	7.278
11	3	1	3.532	61	3	6	7.418
12	0	3	3.615	62	8	3	7.421
13	5	0	3.615	63	0	8	7.432
14	2	2	3.799	64	13	0	7.436
15	4	1	3.911	65	5	5	7.511
16	1	3	4.228	66	10	2	7.511
17	6	0	4.228	67	2	7	7.724
18	3	2	4.352	68	7	4	7.725
19	0	4	4.480	69	12	1	7.725
20	5	1	4.480	70	4	6	7.859
21	2	3	4.598	71	9	3	7.867
22	7	0	4.598	72	1	8	7.889
23	4	2	4.814	73	14	0	7.895
24	1	4	5.014	74	6	5	8.012
25	6	1	5.014	75	11	2	8.012
26	3	3	5.144	76	3	7	8.154
27	8	0	5.144	77	0	9	8.179
28	0	5	5.221	78	8	4	8.179
29	5	2	5.239	79	13	1	8.246
30	2	4	5.477	80	5	6	8.246
31	7	1	5.477	81	2	8	8.454
32	4	3	5.598	82	10	3	8.454
33	9	0	5.599	83	15	0	8.454
34	1	5	5.781	84	7	5	8.455
35	6	2	5.782	85	4	7	8.592
36	3	4	5.915	86	12	2	8.596
37	8	1	5.917	87	1	9	8.737
38	0	6	5.963	88	9	4	8.750
39	5	3	6.056	89	6	6	8.751
40	10	0	6.058	90	14	1	8.752
41	2	5	6.238	91	3	8	8.882
42	7	2	6.238	92	11	3	8.882
43	4	4	6.364	93	0	10	8.891
44	9	1	6.366	94	8	5	8.891
45	1	6	6.535	95	16	0	8.891
46	6	3	6.535	96	5	7	8.973
47	11	0	6.535	97	13	2	8.973
48	3	5	6.672	98	2	9	9.178
49	8	2	6.674	99	10	4	9.178
50	0	7	6.698	100	7	6	9.178

TABLE 4

ALPHA= .05 A1=.10 K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	2.695	51	2	2	5.705
2	1	0	2.700	52	16	1	5.705
3	2	0	2.735	53	32	0	5.705
4	3	0	2.787	54	3	2	5.757
5	4	0	2.849	55	17	1	5.757
6	5	0	2.917	56	33	0	5.757
7	6	0	2.991	57	4	2	5.819
8	7	0	3.068	58	18	1	5.819
9	8	0	3.148	59	34	0	5.819
10	9	0	3.230	60	5	2	5.889
11	10	0	3.314	61	19	1	5.889
12	11	0	3.400	62	35	0	5.889
13	12	0	3.487	63	6	2	5.963
14	13	0	3.575	64	20	1	5.963
15	14	0	3.664	65	36	0	5.963
16	15	0	3.754	66	7	2	6.041
17	16	0	3.844	67	21	1	6.041
18	17	0	3.936	68	37	0	6.041
19	0	1	4.269	69	8	2	6.121
20	1	1	4.275	70	22	1	6.121
21	18	0	4.275	71	38	0	6.121
22	2	1	4.309	72	9	2	6.204
23	19	0	4.309	73	23	1	6.204
24	3	1	4.360	74	39	0	6.204
25	20	0	4.360	75	10	2	6.289
26	4	1	4.422	76	24	1	6.289
27	21	0	4.422	77	40	0	6.289
28	5	1	4.491	78	11	2	6.376
29	22	0	4.491	79	25	1	6.376
30	6	1	4.565	80	41	0	6.376
31	23	0	4.565	81	12	2	6.464
32	7	1	4.643	82	26	1	6.464
33	24	0	4.643	83	0	3	6.978
34	8	1	4.723	84	42	0	6.978
35	25	0	4.723	85	13	2	6.978
36	9	1	4.806	86	27	1	6.978
37	26	0	4.806	87	1	3	6.984
38	10	1	4.891	88	43	0	6.984
39	27	0	4.891	89	14	2	6.984
40	11	1	4.977	90	28	1	6.984
41	28	0	4.977	91	2	3	7.018
42	12	1	5.065	92	15	2	7.018
43	29	0	5.065	93	44	0	7.018
44	13	1	5.154	94	29	1	7.018
45	0	2	5.666	95	3	3	7.069
46	30	0	5.666	96	16	2	7.069
47	14	1	5.666	97	45	0	7.069
48	1	2	5.672	98	30	1	7.069
49	15	1	5.672	99	4	3	7.132
50	31	0	5.672	100	17	2	7.132

TABLE 4 (CONT.)

ALPHA= .05   A1=.20   K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	2.396	51	4	3	6.664
2	1	0	2.425	52	15	1	6.664
3	2	0	2.534	53	21	0	6.664
4	3	0	2.677	54	10	2	6.666
5	4	0	2.840	55	5	3	6.850
6	5	0	3.015	56	0	4	7.323
7	6	0	3.198	57	16	1	7.323
8	0	1	3.795	58	22	0	7.323
9	7	0	3.795	59	11	2	7.323
10	1	1	3.825	60	6	3	7.323
11	8	0	3.825	61	1	4	7.365
12	2	1	3.934	62	17	1	7.411
13	9	0	3.934	63	23	0	7.464
14	3	1	4.081	64	12	2	7.464
15	10	0	4.081	65	7	3	7.464
16	4	1	4.248	66	2	4	7.464
17	11	0	4.248	67	18	1	7.464
18	5	1	4.429	68	13	2	7.464
19	0	2	5.037	69	24	0	7.464
20	12	0	5.037	70	8	3	7.464
21	6	1	5.037	71	3	4	7.614
22	1	2	5.067	72	19	1	7.614
23	7	1	5.067	73	14	2	7.614
24	13	0	5.067	74	25	0	7.614
25	2	2	5.176	75	9	3	7.614
26	8	1	5.176	76	4	4	7.787
27	14	0	5.176	77	20	1	7.787
28	3	2	5.325	78	15	2	7.787
29	9	1	5.325	79	26	0	7.787
30	15	0	5.325	80	10	3	7.790
31	4	2	5.494	81	5	4	7.975
32	10	1	5.495	82	0	5	8.410
33	16	0	5.495	83	21	1	8.410
34	5	2	5.677	84	16	2	8.410
35	0	3	6.202	85	27	0	8.410
36	11	1	6.202	86	11	3	8.410
37	17	0	6.202	87	6	4	8.410
38	6	2	6.202	88	1	5	8.420
39	1	3	6.298	89	22	1	8.499
40	12	1	6.343	90	17	2	8.552
41	18	0	6.343	91	28	0	8.552
42	7	2	6.343	92	12	3	8.552
43	2	3	6.343	93	7	4	8.552
44	13	1	6.343	94	2	5	8.552
45	19	0	6.343	95	23	1	8.552
46	8	2	6.343	96	18	2	8.552
47	3	3	6.493	97	13	3	8.552
48	14	1	6.493	98	29	0	8.552
49	20	0	6.493	99	8	4	8.552
50	9	2	6.493	100	3	5	8.703

TABLE 4 (CONT.)

ALPHA= .05 A1=.25 K=2

I	X1	X2	THETA*
1	0	0	2.247
2	1	0	2.301
3	2	0	2.464
4	3	0	2.671
5	4	0	2.901
6	0	1	3.557
7	5	0	3.557
8	1	1	3.613
9	6	0	3.613
10	2	1	3.781
11	7	0	3.781
12	3	1	3.997
13	8	0	4.001
14	4	1	4.239
15	0	2	4.720
16	9	0	4.720
17	5	1	4.720
18	1	2	4.777
19	10	0	4.777
20	2	2	4.946
21	6	1	4.948
22	11	0	4.948
23	3	2	5.160
24	7	1	5.172
25	0	3	5.815
26	12	0	5.815
27	4	2	5.815
28	8	1	5.815
29	1	3	5.871
30	5	2	5.871
31	13	0	5.871
32	9	1	5.871
33	2	3	6.041
34	6	2	6.044
35	14	0	6.044
36	10	1	6.044
37	3	3	6.257
38	7	2	6.274
39	11	1	6.274
40	15	0	6.274
41	0	4	6.864
42	4	3	6.864
43	8	2	6.864
44	12	1	6.864
45	16	0	6.864
46	1	4	6.921
47	5	3	6.921
48	9	2	6.921
49	13	1	6.921
50	17	0	6.921

I	X1	X2	THETA*
51	2	4	7.092
52	6	3	7.096
53	10	2	7.096
54	14	1	7.096
55	18	0	7.096
56	3	4	7.309
57	7	3	7.332
58	11	2	7.409
59	0	5	7.884
60	15	1	7.884
61	19	0	7.884
62	4	4	7.884
63	8	3	7.884
64	12	2	7.884
65	1	5	7.941
66	16	1	7.941
67	20	0	7.941
68	5	4	7.941
69	9	3	7.941
70	13	2	7.941
71	2	5	8.113
72	17	1	8.113
73	21	0	8.113
74	6	4	8.118
75	10	3	8.140
76	14	2	8.175
77	3	5	8.175
78	18	1	8.175
79	22	0	8.221
80	7	4	8.367
81	0	6	8.882
82	11	3	8.882
83	15	2	8.882
84	4	5	8.882
85	19	1	8.882
86	23	0	8.882
87	8	4	8.882
88	1	6	8.938
89	12	3	8.938
90	16	2	8.938
91	5	5	8.938
92	20	1	8.938
93	9	4	8.938
94	24	0	8.938
95	2	6	9.111
96	13	3	9.111
97	17	2	9.111
98	6	5	9.116
99	21	1	9.116
100	10	4	9.116

TABLE 4 (CONT.)

ALPHA= .05   A1=.30   K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	2.097	51	15	0	7.358
2	1	0	2.188	52	1	5	7.455
3	2	0	2.426	53	4	4	7.459
4	3	0	2.714	54	7	3	7.459
5	0	1	3.321	55	10	2	7.459
6	4	0	3.321	56	13	1	7.459
7	1	1	3.416	57	2	5	7.695
8	5	0	3.416	58	16	0	7.695
9	2	1	3.666	59	5	4	7.745
10	6	0	3.742	60	8	3	7.777
11	0	2	4.406	61	11	2	7.804
12	3	1	4.406	62	0	6	8.289
13	1	2	4.501	63	14	1	8.289
14	7	0	4.501	64	3	5	8.289
15	4	1	4.504	65	17	0	8.289
16	2	2	4.740	66	6	4	8.289
17	5	1	4.765	67	9	3	8.289
18	8	0	4.765	68	12	2	8.289
19	0	3	5.427	69	1	6	8.386
20	3	2	5.427	70	15	1	8.386
21	6	1	5.427	71	4	5	8.390
22	9	0	5.427	72	18	0	8.390
23	1	3	5.522	73	7	4	8.390
24	4	2	5.525	74	10	3	8.390
25	7	1	5.525	75	2	6	8.626
26	10	0	5.525	76	13	2	8.626
27	2	3	5.761	77	16	1	8.626
28	5	2	5.796	78	5	5	8.683
29	8	1	5.798	79	19	0	8.683
30	11	0	5.798	80	8	4	8.728
31	0	4	6.407	81	0	7	9.203
32	3	3	6.407	82	11	3	9.203
33	6	2	6.407	83	3	6	9.203
34	9	1	6.407	84	14	2	9.203
35	12	0	6.407	85	17	1	9.203
36	1	4	6.503	86	6	5	9.203
37	4	3	6.506	87	20	0	9.203
38	7	2	6.506	88	9	4	9.203
39	10	1	6.506	89	1	7	9.300
40	13	0	6.506	90	12	3	9.300
41	2	4	6.742	91	4	6	9.305
42	5	3	6.785	92	15	2	9.305
43	8	2	6.800	93	7	5	9.305
44	11	1	6.809	94	18	1	9.305
45	14	0	6.810	95	21	0	9.305
46	0	5	7.358	96	10	4	9.305
47	3	4	7.358	97	2	7	9.541
48	6	3	7.358	98	13	3	9.541
49	9	2	7.358	99	5	6	9.604
50	12	1	7.358	100	16	2	9.604

TABLE 4 (CONT.)

ALPHA= .05 A1=1/3 K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	1.997	51	9	2	7.457
2	1	0	2.125	52	0	6	7.894
3	2	0	2.427	53	14	0	7.894
4	0	1	3.162	54	7	3	7.894
5	3	0	3.162	55	5	4	7.894
6	1	1	3.295	56	12	1	7.894
7	4	0	3.300	57	3	5	7.894
8	2	1	3.599	58	10	2	7.894
9	0	2	4.197	59	1	6	8.035
10	5	0	4.197	60	8	3	8.035
11	3	1	4.197	61	15	0	8.035
12	1	2	4.333	62	6	4	8.036
13	6	0	4.333	63	13	1	8.036
14	4	1	4.341	64	4	5	8.175
15	2	2	4.638	65	11	2	8.177
16	7	0	4.639	66	2	6	8.344
17	0	3	5.168	67	9	3	8.345
18	5	1	5.168	68	16	0	8.345
19	3	2	5.168	69	0	7	8.765
20	8	0	5.168	70	7	4	8.765
21	1	3	5.306	71	14	1	8.765
22	6	1	5.306	72	5	5	8.765
23	4	2	5.385	73	12	2	8.765
24	9	0	5.389	74	3	6	8.765
25	2	3	5.613	75	10	3	8.765
26	7	1	5.616	76	17	0	8.765
27	0	4	6.102	77	1	7	8.906
28	5	2	6.102	78	8	4	8.906
29	10	0	6.102	79	15	1	8.906
30	3	3	6.102	80	6	5	8.907
31	8	1	6.102	81	13	2	8.915
32	1	4	6.241	82	4	6	9.059
33	6	2	6.241	83	11	3	9.061
34	11	0	6.241	84	18	0	9.061
35	4	3	6.288	85	2	7	9.134
36	9	1	6.359	86	9	4	9.134
37	2	4	6.548	87	0	8	9.142
38	0	5	7.009	88	16	1	9.144
39	7	2	7.009	89	7	5	9.249
40	12	0	7.009	90	14	2	9.287
41	5	3	7.009	91	5	6	9.344
42	10	1	7.009	92	12	3	9.344
43	3	4	7.009	93	3	7	9.568
44	1	5	7.148	94	19	0	9.568
45	8	2	7.148	95	10	4	9.631
46	13	0	7.148	96	1	8	9.631
47	6	3	7.148	97	17	1	9.631
48	4	4	7.274	98	8	5	9.631
49	11	1	7.274	99	15	2	9.765
50	2	5	7.457	100	6	6	9.765

TABLE 4 (CONT.)

ALPHA= .05   A1=.40   K=2

I	X1	X2	THETA*	I	X1	X2	THETA*
1	0	0	1.797	51	5	4	7.556
2	1	0	2.056	52	10	1	7.556
3	0	1	2.846	53	2	6	7.651
4	2	0	2.846	54	7	3	7.724
5	1	1	3.080	55	12	0	7.724
6	3	0	3.158	56	4	5	7.908
7	0	2	3.777	57	9	2	7.911
8	2	1	3.777	58	1	7	8.106
9	4	0	3.777	59	6	4	8.107
10	1	2	4.004	60	11	1	8.107
11	3	1	4.103	61	3	6	8.237
12	5	0	4.209	62	8	3	8.238
13	0	3	4.209	63	13	0	8.238
14	2	2	4.410	64	0	8	8.259
15	4	1	4.537	65	5	5	8.371
16	6	0	4.537	66	10	2	8.375
17	1	3	4.876	67	2	7	8.572
18	3	2	4.985	68	7	4	8.573
19	5	1	5.086	69	12	1	8.573
20	0	4	5.091	70	4	6	8.579
21	7	0	5.206	71	9	3	8.579
22	2	3	5.206	72	14	0	8.579
23	4	2	5.464	73	1	8	8.878
24	1	4	5.713	74	6	5	8.878
25	6	1	5.713	75	11	2	8.878
26	3	3	5.831	76	3	7	9.010
27	8	0	5.831	77	8	4	9.013
28	0	5	5.933	78	0	9	9.065
29	5	2	5.933	79	13	1	9.074
30	2	4	6.108	80	5	6	9.114
31	7	1	6.130	81	10	3	9.155
32	4	3	6.213	82	2	8	9.339
33	9	0	6.213	83	15	0	9.339
34	1	5	6.527	84	7	5	9.339
35	6	2	6.528	85	12	2	9.339
36	3	4	6.650	86	4	7	9.344
37	8	1	6.651	87	9	4	9.457
38	0	6	6.774	88	1	9	9.461
39	5	3	6.774	89	14	1	9.461
40	10	0	6.775	90	6	6	9.591
41	2	5	6.903	91	11	3	9.640
42	7	2	6.914	92	3	8	9.775
43	4	4	7.113	93	16	0	9.775
44	9	1	7.151	94	8	5	9.778
45	1	6	7.324	95	0	10	9.814
46	6	3	7.324	96	13	2	9.848
47	11	0	7.324	97	5	7	9.877
48	3	5	7.451	98	10	4	9.877
49	8	2	7.498	99	2	9	10.095
50	0	7	7.556	100	15	1	10.096